A Potential Theory of General Spatially-Coupled Systems via a Continuum Approximation

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***Abstract*—This paper analyzes general spatially-coupled (SC) vector systems with multi-dimensional coupling. A continuum approximation is used to derive potential functions that characterize the performance of the SC systems. For one-dimensional and two-dimensional coupling, it is shown that, if the boundary of the SC systems is fixed to the unique stable solution that minimizes the potential over all stationary solutions, the systems can approach the optimal performance as the number of coupled systems tends to infinity. Furthermore, a physical intuition is presented for this result on the basis of the conservation law of mechanical energy in Newton mechanics.**

I. Introduction

Kudekar et al. [1] proved that spatial coupling can improve the belief-propagation (BP) performance of low-density parity-check (LDPC) codes up to the maximum-a-posteriori (MAP) performance. This phenomenon, called *threshold saturation*, has been observed in many other spatially-coupled (SC) systems, such as the MacKay-Neal and Hsu-Anastasopoulos codes [2], code-division multiple-access (CDMA) [3]–[5], compressed sensing [6], [7], and physical models [8]. Thus, threshold saturation via spatial coupling is believed to be a universal phenomenon.

In order to prove the universality of threshold saturation, theoretical analyses have been performed for general SC systems with one-dimensional coupling [9]–[11]. The methodologies are classified into those based on potential functions [9], [10] and on extrinsic information transfer (EXIT) functions [11]. Potential functions were also used for the analysis of threshold saturation in [7], [8]. The potential-based methodology has the advantage that the analysis of the BP performance is simplified. Yedla et al. [10] defined a potential function to specify the BP performance for general SC systems. However, they presented no derivation of the potential function. The main purpose of this paper is to present a systematic derivation of the potential function.

The derivation is based on the continuum approximation used in [4], [9]. The previous analysis for SC scalar systems with one-dimensional coupling is generalized to the case of SC vector systems with multi-dimensional coupling. Hereafter, general SC vector systems with multi-dimensional coupling is simply referred to as generalized SC (GSC) systems.

The main contributions of this paper are summarized as follows: (i) The potential function defined in [10] is systematically derived via the continuum approximation. (ii) Intuitive insights for the threshold-saturation phenomenon are presented on the basis of classical mechanics. (iii) Two-dimensional coupling is shown to provide the same improvement of the BP performance as one-dimensional coupling.

II. System Model

A. Notation

For integers i and j ($> i$), $[i:j]$ denotes the set $\{i, i+1, \ldots, j\}$ of lattice points. The symbols δ_{ab}, δ_a^b, δ^{ab} represent the Kronecker delta. As defined below, the state of a GSC system with K-dimensional coupling is represented by two N-dimensional vector fields $\boldsymbol{u}(\boldsymbol{x}, t) = \{u^a(\boldsymbol{x}, t) : a = 1, \ldots, N\}$ and $\boldsymbol{v}(\boldsymbol{x}, t) = \{v_a(\boldsymbol{x}, t) : a = 1, \ldots, N\}$ on $\mathbb{R}^K \times [0, \infty)$. Roman alphabet is used for the indices of the elements of the vector fields, whereas Greek alphabet is for the spatial vector $\boldsymbol{x} = \{x^\alpha : \alpha = 1, \ldots, K\}$. The differential operators $\partial/\partial u^a$ and $\partial/\partial v_b$ are abbreviated to ∂_a and ∂^b, respectively. The gradients $\{\partial_a : a = 1, \ldots, N\}$ and $\{\partial^a : a = 1, \ldots, N\}$ are denoted by ∇ and $\tilde{\nabla}$, respectively. One the other hand, $\partial/\partial \boldsymbol{x}$ represents the gradient $\{\partial/\partial x^\alpha : \alpha = 1, \ldots, K\}$ for the spatial variables. Furthermore, the Einstein summation convention is used: When an index appears twice in a single term, the summation is taken over all values of the index. For example, $u^a v_a = \sum_{a=1}^N u^a v_a$.

B. Uncoupled System

For two scalar fields F and G on $\mathbb{R}^N$, let $\mathcal{D} \subset \mathbb{R}^N$ and $\tilde{\mathcal{D}} \subset \mathbb{R}^N$ denote the images of the gradients $\tilde{\nabla} F$ and ∇G, respectively. The dimension N corresponds to the number of parameters required for describing asymptotic performance of the BP algorithm for a system with no coupling. We assume that asymptotic performance is characterized by the coupled density-evolution (DE) equations with respect to $\boldsymbol{u}(t) = \{u^a(t) : a = 1, \ldots, N\}$ and $\boldsymbol{v}(t) = \{v_a(t) : a = 1, \ldots, N\}$,

$$\boldsymbol{u}(t+1) = \tilde{\nabla} F(\boldsymbol{v}(t)) \in \mathcal{D}, \tag{1}$$

$$\boldsymbol{v}(t) = \nabla G(\boldsymbol{u}(t)) \in \tilde{\mathcal{D}}, \tag{2}$$

where the time index t corresponds to the number of iterations for the BP algorithm. The asymptotic performance in iteration t is assumed to be characterized by a deterministic scalar

function $P(\cdot)$ of the state $\boldsymbol{u}(t)$ that starts from an appropriate initial state.

Assumption 1. *The two scalar fields F and G are thrice continuously differentiable on $\tilde{\mathcal{D}}$ and $\mathcal{D}$, respectively. Let $\mathcal{S} \times \tilde{\mathcal{S}} \subset \mathcal{D} \times \tilde{\mathcal{D}}$ denote the set of all fixed-points (FPs) $(\boldsymbol{u}, \boldsymbol{v})$ for the DE equations (1) and (2). The Hesse matrix $f^{ab}(\boldsymbol{v}) = \partial^a \partial^b F(\boldsymbol{v})$ is non-singular for all $\boldsymbol{v} \in \tilde{\mathcal{D}}_0 = \tilde{\mathcal{D}} \backslash \tilde{\mathcal{S}}$, and the quadratic form $y_a y_b f^{ab}(\boldsymbol{v})$ is bounded below for all $\{y_a\} \in \tilde{\mathcal{D}}$ and $\boldsymbol{v} \in \tilde{\mathcal{D}}$. On the other hand, the Hesse matrix $g_{ab}(\boldsymbol{u}) = \partial_a \partial_b G(\boldsymbol{u})$ are positive (resp. non-negative) definite for $\boldsymbol{u} \in \mathcal{D}_0 = \mathcal{D} \backslash \mathcal{S}$ (resp. $\boldsymbol{u} \in \mathcal{D}$).*

Examples included in our framework are MN and HA codes [2]. Note that Assumption 1 is different from that postulated in [10]: The positive-definiteness of the Hesse matrix $g_{ab}(\boldsymbol{u})$ is assumed in this paper, whereas the positivity of its elements is postulated in [10].

In order to investigate the convergence property of the state $\boldsymbol{u}(t)$, we define two potential functions, one of which was originally defined in [10]. They are equivalent to the so-called trial entropy for LDPC codes [10] and to the free energy for CDMA [4], characterizing the MAP performance.

Definition 1 (Potential)**.** *Let $D(\boldsymbol{u}, \boldsymbol{v})$ denote the divergence*

$$D(\boldsymbol{u}, \boldsymbol{v}) = G(\boldsymbol{u}) + F(\boldsymbol{v}) - u^a v_a. \tag{3}$$

The potential function $V(\boldsymbol{u})$ is defined as

$$V(\boldsymbol{u}) = -D(\boldsymbol{u}, \nabla G(\boldsymbol{u})). \tag{4}$$

On the other hand, the dual potential function $\tilde{V}(\boldsymbol{v})$ is defined by

$$\tilde{V}(\boldsymbol{v}) = -D(\tilde{\nabla} F(\boldsymbol{v}), \boldsymbol{v}). \tag{5}$$

Assumption 2. *The potential functions (4) and (5) are bounded below.*

Assumption 2 is a sufficient condition for guaranteeing the convergence of the state $\boldsymbol{u}(t)$ toward a FP $\boldsymbol{u}^*$ as $t \to \infty$. The following proposition implies that the two potential functions have the same information about FPs.

Proposition 1. *The 2-tuple $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ is a FP of the DE equations (1) and (2) if and only if $\boldsymbol{u}^*$ (resp. $\boldsymbol{v}^*$) is a stationary solution of the potential (4) (resp. (5)). Furthermore, $V(\boldsymbol{u}^*) = \tilde{V}(\boldsymbol{v}^*)$ at any FP $(\boldsymbol{u}^*, \boldsymbol{v}^*)$.*

Proof: Calculating the gradient of the potential (4) with (3) yields

$$\partial_a V(\boldsymbol{u}) = g_{ab}(\boldsymbol{u})(u^b - \partial^b F(\nabla G(\boldsymbol{u})). \tag{6}$$

Since the Hesse matrix $\{g_{ab}(\boldsymbol{u})\}$ of G is non-singular except for the FPs $\boldsymbol{u}^* \in \mathcal{S}$, $\partial_a V(\boldsymbol{u}^*) = 0$ is equivalent to $\boldsymbol{u}^* = \tilde{\nabla} F(\boldsymbol{v}^*)$ with $\boldsymbol{v}^* = \nabla G(\boldsymbol{u}^*)$. Similarly, we find that $\partial^a \tilde{V}(\boldsymbol{v}^*) = 0$ is equivalent to the condition that $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ is a FP. Furthermore, we have

$$\tilde{V}(\nabla G(\boldsymbol{u}^*)) = -D(\tilde{\nabla} F(\nabla G(\boldsymbol{u}^*)), \nabla G(\boldsymbol{u}^*)) = V(\boldsymbol{u}^*), \tag{7}$$

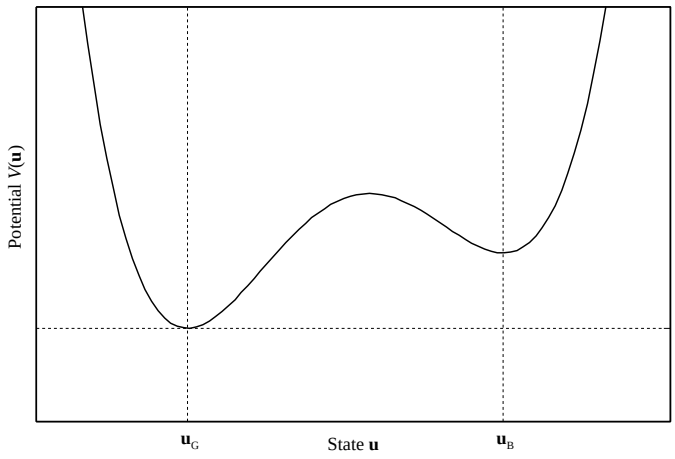

Fig. 1. Shape of the potential postulated in this paper.

where $\boldsymbol{u}^* = \tilde{\nabla} F(\nabla G(\boldsymbol{u}^*))$ has been used. This implies $\tilde{V}(\boldsymbol{v}^*) = V(\boldsymbol{u}^*)$. ■

Let us investigate the convergence property of the state. We use the approximation $\boldsymbol{u}(t+1) - \boldsymbol{u}(t) \approx d\boldsymbol{u}/dt$ for (1) to obtain

$$\frac{du^a}{dt} \approx -\{u^a - \partial^a F(\nabla G(\boldsymbol{u}))\} = -g^{ab}(\boldsymbol{u}) \partial_b V(\boldsymbol{u}), \tag{8}$$

for $\boldsymbol{u} \in \mathcal{D}_0$, where we have used (6). In (8), $\{g^{ab}(\boldsymbol{u})\}$ denotes the inverse matrix of the Hesse matrix $\{g_{ab}(\boldsymbol{u})\}$. It is straightforward to confirm that the potential (4) is a Lyapunov function for the continuous-time system (8).

$$\frac{dV}{dt}(\boldsymbol{u}(t)) = \partial_a V \frac{du^a}{dt} = -\partial_a V(\boldsymbol{u}(t)) \partial_b V(\boldsymbol{u}(t)) g^{ab}(\boldsymbol{u}(t)), \tag{9}$$

which is negative for all $\boldsymbol{u}(t) \in \mathcal{D}_0$, and zero only when $\boldsymbol{u}(t)$ is at a stationary solution of the potential (4). Since the potential (4) is bounded below, this observation implies that the state $\boldsymbol{u}(t)$ for the continuous-time system (8) converges to a stationary solution of the potential (4) as $t \to \infty$.

C. General Spatially-Coupled System

Let us consider the case of double-well potential shown in Fig. 1. The potential $V(\boldsymbol{u}_{\mathrm{G}})$ at the left stable solution $\boldsymbol{u}_{\mathrm{G}}$ is lower than that at the right stable solution $\boldsymbol{u}_{\mathrm{B}}$. Furthermore, the performance $P(\boldsymbol{u}_{\mathrm{G}})$ is assumed to be better than $P(\boldsymbol{u}_{\mathrm{B}})$. Note that there is no relationship between $P(\boldsymbol{u})$ and $V(\boldsymbol{u})$. We hereafter refer to $\boldsymbol{u}_{\mathrm{G}}$ and $\boldsymbol{u}_{\mathrm{B}}$ as the good and bad solutions, respectively. The state $\boldsymbol{u}(t)$ for the BP algorithm should converge to the bad stable solution $\boldsymbol{u}_{\mathrm{B}}$, since the BP algorithm starts from a bad initial state. On the other hand, the MAP algorithm may achieve the better performance $P(\boldsymbol{u}_{\mathrm{G}})$, which implies the suboptimality of the BP algorithm. Spatial coupling is a method for helping the state climb the potential barrier and arrive at the good stable solution $\boldsymbol{u}_{\mathrm{G}}$.

Suppose that a GSC system with K-dimensional coupling of size L and width W is characterized by the DE equations,

$$\boldsymbol{u}\left(\frac{\boldsymbol{l}}{L}, t+1\right) = \left\langle \tilde{\nabla} F\left(\boldsymbol{v}\left(\frac{\boldsymbol{l}+\boldsymbol{m}}{L}, t\right)\right)\right\rangle_{\boldsymbol{m}}, \tag{10}$$

$$\boldsymbol{v}\left(\frac{\boldsymbol{l}}{L},t\right)=\left\langle \nabla G\left(\boldsymbol{u}\left(\frac{\boldsymbol{l}-\boldsymbol{m}}{L},t\right)\right)\right\rangle_{\boldsymbol{m}}, \tag{11}$$

for all spatial positions $\boldsymbol{l}\in[-L+1:L-1]^K$, with

$$\langle f(\boldsymbol{m})\rangle_{\boldsymbol{m}}=\frac{1}{(2W+1)^K}\sum_{\boldsymbol{m}\in[-W:W]^K} f(\boldsymbol{m}). \tag{12}$$

We impose the boundary condition $\boldsymbol{u}(\boldsymbol{l},t)=\boldsymbol{u}_{\rm G}$ for all $\boldsymbol{l}\in\mathbb{Z}^K\backslash[-L+1:L-1]^K$. This condition corresponds to informing the detector about the true solutions for the systems at the boundary in advance.

The idea of spatial coupling is explained as follows: The correct information at the boundary may spread over the whole system via coupling, regardless of size L. Since the influence of the boundary is negligible as $L\to\infty$, the loss due to informing the detector about the true solutions is also negligible.

III. Continuum Approximation

In order to investigate the convergence property of the DE equations (10) and (11), we use the continuum approximation with respect to the spatial variable $\boldsymbol{x}=\boldsymbol{l}/L$ as $L\to\infty$. For notational convenience, the argument $(\boldsymbol{x},t)$ of functions is omitted.

Lemma 1. *Let*

$$M=\frac{1}{L^2(2W+1)}\sum_{m=-W}^{W} m^2. \tag{13}$$

As $L\to\infty$, the DE equations (1) and (2) are respectively approximated by the spatially continuous equations

$$u^a(\boldsymbol{x},t+1)=\partial^a F(\boldsymbol{v})+\frac{M}{2}\sum_{\alpha=1}^{K}\left(f^{ab}(\boldsymbol{v})\frac{\partial^2 v_b}{\partial {x^\alpha}^2}+\partial^a\partial^b\partial^c F(\boldsymbol{v})\frac{\partial v_b}{\partial x^\alpha}\frac{\partial v_c}{\partial x^\alpha}\right)+o(L^2), \tag{14}$$

$$v_a=\partial_a G(\boldsymbol{u})+\frac{M}{2}\sum_{\alpha=1}^{K}\left(g_{ab}(\boldsymbol{u})\frac{\partial^2 u^b}{\partial {x^\alpha}^2}+\partial_a\partial_b\partial_c G(\boldsymbol{u})\frac{\partial u^b}{\partial x^\alpha}\frac{\partial u^c}{\partial x^\alpha}\right)+o(L^2). \tag{15}$$

Proof: The proof is based on a straightforward Taylor expansion with respect to $1/L$, and therefore omitted. ∎

Note that cross terms with respect to the spatial variables vanish because of $\langle \boldsymbol{m}\rangle_{\boldsymbol{m}}=\boldsymbol{0}$. In terms of differential geometry [12], the variable $\boldsymbol{u}$ may be regarded as the coordinates associated with a local coordinate system in a differential manifold with an affine connection. The coefficient $\partial_a\partial_b\partial_c G(\boldsymbol{u})$ in (15) corresponds to the coefficient of the affine connection for the local coordinate system. The same interpretation holds for $\boldsymbol{v}$ and $\partial^a\partial^b\partial^c F(\boldsymbol{v})$. A direct calculation implies that the two manifolds for $\boldsymbol{u}$ and $\boldsymbol{v}$ are flat, i.e. the torsion and curvature tensors are everywhere zero. In order to simplify the expressions of (14) and (15), we use affine coordinate systems for which the coefficients of the affine connections vanish everywhere.

Theorem 1. *Consider the affine coordinates $(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}})$ that is defined by the twice continuously differentiable mapping from $(\boldsymbol{u},\boldsymbol{v})\in\mathcal{D}_0\times\tilde{\mathcal{D}}_0$ onto $(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}})\in\mathcal{D}_0\times\tilde{\mathcal{D}}_0$,*

$$(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}})=(\tilde{\nabla}F(\boldsymbol{v}),\nabla G(\boldsymbol{u})). \tag{16}$$

Fix $(\boldsymbol{x},t)$ and suppose that $(\boldsymbol{u},\boldsymbol{v})\in\mathcal{D}_0\times\tilde{\mathcal{D}}_0$ in a neighborhood of $(\boldsymbol{x},t)$. Then, the equations (14) and (15) reduce to the two independent equations with respect to the affine coordinates $\tilde{\boldsymbol{v}}=\{\tilde{v}_a:a=1,\dots,N\}$ and $\tilde{\boldsymbol{u}}=\{\tilde{u}^a:a=1,\dots,N\}$,

$$\Phi^a(\tilde{\boldsymbol{v}}(\boldsymbol{x},t+1))-\Phi^a(\tilde{\boldsymbol{v}}(\boldsymbol{x},t))=-\frac{\partial}{\partial\tilde{v}_a}V(\boldsymbol{\Phi}(\tilde{\boldsymbol{v}}))+M\mathfrak{C}^a(\tilde{\boldsymbol{v}})+o(L^2), \tag{17}$$

$$\Psi_a(\tilde{\boldsymbol{u}}(\boldsymbol{x},t+1))-\Psi_a(\tilde{\boldsymbol{u}}(\boldsymbol{x},t))=-\frac{\partial}{\partial\tilde{u}^a}\tilde{V}(\boldsymbol{\Psi}(\tilde{\boldsymbol{u}}))+M\tilde{\mathfrak{C}}_a(\tilde{\boldsymbol{u}})+o(L^2), \tag{18}$$

with

$$\mathfrak{C}^a(\tilde{\boldsymbol{v}})=\sum_{\alpha=1}^{K}\left(f^{ab}(\tilde{\boldsymbol{v}})\frac{\partial^2\tilde{v}_b}{\partial {x^\alpha}^2}+\frac{1}{2}\frac{\partial f^{ab}}{\partial v_c}(\tilde{\boldsymbol{v}})\frac{\partial\tilde{v}_b}{\partial x^\alpha}\frac{\partial\tilde{v}_c}{\partial x^\alpha}\right), \tag{19}$$

$$\tilde{\mathfrak{C}}_a(\tilde{\boldsymbol{u}})=\sum_{\alpha=1}^{K}\left(g_{ab}(\tilde{\boldsymbol{u}})\frac{\partial^2\tilde{u}^b}{\partial {x^\alpha}^2}+\frac{1}{2}\frac{\partial g_{ab}}{\partial u_c}(\tilde{\boldsymbol{u}})\frac{\partial\tilde{u}^b}{\partial x^\alpha}\frac{\partial\tilde{u}^c}{\partial x^\alpha}\right). \tag{20}$$

In the first terms on the right-hand sides (RHSs) of (17) and (18), the potential functions are given by (4) and (5), respectively. Furthermore, $\boldsymbol{\Phi}(\tilde{\boldsymbol{v}})=\{\Phi^a(\tilde{\boldsymbol{v}}):a=1,\dots,N\}$ and $\boldsymbol{\Psi}(\tilde{\boldsymbol{u}})=\{\Psi_a(\tilde{\boldsymbol{u}}):a=1,\dots,N\}$ denote the inverse mapping of $\tilde{\boldsymbol{v}}=\nabla G(\boldsymbol{u})$ and $\tilde{\boldsymbol{u}}=\tilde{\nabla}F(\boldsymbol{v})$, respectively.

The first terms on the RHSs of (17) and (18) are potential terms determined by the properties of the corresponding uncoupled system, whereas the second terms represent the effect of spatial coupling.

Proof: The change of coordinates (16) transforms the second term on the RHS of (15) into

$$v_a=\partial_a G(\boldsymbol{u})+\frac{M}{2}\sum_{\alpha=1}^{K}\left(\frac{\partial^2\tilde{v}_a}{\partial {x^\alpha}^2}+\Gamma_a^{bc}\frac{\partial\tilde{v}_b}{\partial x^\alpha}\frac{\partial\tilde{v}_c}{\partial x^\alpha}\right)+o(L^2), \tag{21}$$

with

$$\Gamma_a^{bc}=g^{db}(\boldsymbol{u})g^{ec}(\boldsymbol{u})\partial_a\partial_d\partial_e G(\boldsymbol{u})+g_{ad}(\boldsymbol{u})\frac{\partial}{\partial\tilde{v}_c}g^{db}(\boldsymbol{u}). \tag{22}$$

Using a formula obtained by differentiating the identity

$$g_{ad}(\boldsymbol{u})g^{db}(\boldsymbol{u})=\delta_a^b, \tag{23}$$

with respect to $\tilde{v}_c$, we find

$$\Gamma_a^{bc}=g^{db}(\boldsymbol{u})g^{ec}(\boldsymbol{u})\partial_a\partial_d\partial_e G(\boldsymbol{u})-\frac{\partial g_{ad}}{\partial\tilde{v}_c}(\boldsymbol{u})g^{db}(\boldsymbol{u})=0, \tag{24}$$

for all $\boldsymbol{u}\in\mathcal{D}_0$, which implies that $\tilde{\boldsymbol{v}}$ is affine coordinates. In the derivation of (24), we have used the chain rule

$$\frac{\partial g_{ad}}{\partial\tilde{v}_c}(\boldsymbol{u})=\frac{\partial g_{ad}}{\partial u^e}\frac{\partial u^e}{\partial\tilde{v}_c}=\partial_a\partial_d\partial_e G(\boldsymbol{u})g^{ec}(\boldsymbol{u}). \tag{25}$$

Thus, (21) reduces to the simple expression

$$v_a = \tilde{v}_a + \frac{M}{2}\sum_{\alpha=1}^{K}\frac{\partial^2 \tilde{v}_a}{\partial x^{\alpha 2}} + o(L^2), \tag{26}$$

with (16). Similarly, (14) is transformed into

$$u^a(\boldsymbol{x}, t+1) = \partial^a F(\boldsymbol{v}) + \frac{M}{2}\sum_{\alpha=1}^{K}\frac{\partial^2 \tilde{u}^a}{\partial x^{\alpha 2}} + o(L^2), \tag{27}$$

with $\tilde{\boldsymbol{u}} = \{\tilde{u}^a : a = 1, \ldots, N\}$.

We next reduce (26) and (27) to the single equation (17) for $\tilde{\boldsymbol{v}}$. The derivation of (18) is performed in the same manner, and therefore omitted. Expanding the first term on the RHS of (27) with (26) yields

$$\partial^a F(\boldsymbol{v}) = \partial^a F(\tilde{\boldsymbol{v}}) + \frac{M}{2} f^{ab}(\tilde{\boldsymbol{v}}) \sum_{\alpha=1}^{K}\frac{\partial^2 \tilde{v}_b}{\partial x^{\alpha 2}} + o(L^2). \tag{28}$$

We use the chain rule to calculate the second derivative of (28) with respect to x^α up to the order $O(1)$,

$$\begin{aligned}
&\frac{\partial^2}{\partial x^{\alpha 2}}\partial^a F(\boldsymbol{v}) \\
=&\frac{\partial^2}{\partial x^{\alpha 2}}\partial^a F(\tilde{\boldsymbol{v}}) + O(L^{-2}) \\
=&f^{ab}(\tilde{\boldsymbol{v}})\frac{\partial^2 \tilde{v}_b}{\partial x^{\alpha 2}} + \frac{\partial f^{ab}}{\partial v_c}(\tilde{\boldsymbol{v}})\frac{\partial \tilde{v}_b}{\partial x^\alpha}\frac{\partial \tilde{v}_c}{\partial x^\alpha} + O(L^{-2}).
\end{aligned} \tag{29}$$

Substituting (28) and (29) with $\tilde{u}^a = \partial^a F(\boldsymbol{v})$ into (27) yields

$$u^a(\boldsymbol{x}, t+1) - u^a(\boldsymbol{x}, t) = -A^a + M\tilde{\mathfrak{C}}^a(\tilde{\boldsymbol{v}}) + o(L^2), \tag{30}$$

with (19) and $A^a = u^a - \partial^a F(\tilde{\boldsymbol{v}})$.

In order to prove (17), we shall show that A^a is a conservative field with the potential (4). Let $\tilde{\Gamma} \subset \tilde{\mathcal{D}}_0$ denote a smooth curve connecting two points $\tilde{\boldsymbol{v}}_0$ and $\tilde{\boldsymbol{v}}$. When the inverse mapping of $\tilde{\boldsymbol{v}} = \nabla G(\boldsymbol{u})$ maps the curve $\tilde{\Gamma}$ to a curve $\Gamma \subset \mathcal{D}_0$, the line integral of A^a along the curve $\tilde{\Gamma}$ yields

$$\begin{aligned}
\int_{\tilde{\Gamma}} A^a d\tilde{v}_a =& \int_{\Gamma} \left\{u^a - \partial^a F(\nabla G(\boldsymbol{u}))\right\} g_{ab}(\boldsymbol{u}) du^b \\
=& V(\boldsymbol{\Phi}(\tilde{\boldsymbol{v}})) + \mathrm{Const},
\end{aligned} \tag{31}$$

where the potential $V(\boldsymbol{u})$ is given by (4). Expression (31) implies that A^a is a conservative force. ■

IV. Potential Theory

We replace the difference on the left-hand side (LHS) of (17) by the differentiation, with (16), to obtain the partial differential equation (PDE)

$$\frac{\partial \tilde{v}_a}{\partial t} = g_{ab}(\boldsymbol{\Phi}(\tilde{\boldsymbol{v}}))\left\{-\frac{\partial}{\partial \tilde{v}_b}V(\boldsymbol{\Phi}(\tilde{\boldsymbol{v}})) + M\tilde{\mathfrak{C}}^a(\tilde{\boldsymbol{v}})\right\}, \tag{32}$$

with (19), where we have ignored the $o(L^2)$ term. This replacement might be justified except for initial iterations, since the state for large L changes quite slowly as t increases. The boundary condition $\tilde{\boldsymbol{v}}(\boldsymbol{x}, t) = \tilde{\boldsymbol{v}}_{\mathrm{G}} = \nabla G(\boldsymbol{u}_{\mathrm{G}})$ is imposed on the boundary $\partial\mathcal{C}_K$ of the K-dimensional cubic $\mathcal{C}_K = [-1, 1]^K \subset \mathbb{R}^K$, since the boundary condition $\boldsymbol{u}(\boldsymbol{l}, t) = \boldsymbol{u}_{\mathrm{G}}$ has been imposed for all $\boldsymbol{l} \in \mathbb{Z}^K \backslash [-L+1 : L-1]^K$. Note that the state $\tilde{\boldsymbol{v}}_{\mathrm{G}}$ corresponds to the good stable solution of the potential (4), i.e. $V(\boldsymbol{\Phi}(\tilde{\boldsymbol{v}}_{\mathrm{G}})) = V(\boldsymbol{u}_{\mathrm{G}})$.

In order to investigate the convergence property of the PDE (17), we define the energy functional of a vector field $\tilde{\boldsymbol{v}}(\boldsymbol{x})$ on $\mathcal{C}_K$

$$H(\boldsymbol{v}) = \int_{\mathcal{C}_K}\left\{V(\boldsymbol{\Phi}(\tilde{\boldsymbol{v}})) + \frac{M}{2}\sum_{\alpha=1}^{K}\frac{\partial \tilde{v}_a}{\partial x^\alpha}\frac{\partial \tilde{v}_b}{\partial x^\alpha} f^{ab}(\tilde{\boldsymbol{v}})\right\} d\boldsymbol{x}. \tag{33}$$

It is straightforward to find that the PDE (32) is represented by

$$\frac{\partial \tilde{v}_a}{\partial t} = -g_{ab}(\boldsymbol{\Phi}(\tilde{\boldsymbol{v}}))\frac{\delta H}{\delta \tilde{v}_b}(\tilde{\boldsymbol{v}}), \tag{34}$$

where $\delta H/\delta \tilde{v}_a$ denotes the functional derivative of (33) with respect to $\tilde{v}_a$. This expression indicates that the state $\tilde{\boldsymbol{v}}(\boldsymbol{x}, t)$ moves in a direction where the energy functional (33) decreases. In fact, a direct calculation implies that the energy functional (33) is a Lyapunov functional for the dynamical system (32):

$$\frac{dH}{dt}(\tilde{\boldsymbol{v}}) = -\int_{\mathcal{C}_K}\frac{\delta H}{\delta \tilde{v}_a}(\tilde{\boldsymbol{v}})\frac{\delta H}{\delta \tilde{v}_b}(\tilde{\boldsymbol{v}}) g_{ab}(\boldsymbol{\Phi}(\tilde{\boldsymbol{v}}))d\boldsymbol{x} \le 0, \tag{35}$$

where the equality holds only when $\delta H/\delta \tilde{v}_a = 0$ is zero for all a. Here, we have used the non-negative definiteness of the Hesse matrix $\{g_{ab}(\boldsymbol{u})\}$. Furthermore, the energy functional (33) is obviously bounded below. These observations imply that (33) is a Lyapunov functional. Thus, it is guaranteed that the state $\tilde{\boldsymbol{v}}(\boldsymbol{x}, t)$ converges to a stationary solution $\tilde{\boldsymbol{v}}(\boldsymbol{x})$ as $t \to \infty$. Note that the convergence of $\tilde{\boldsymbol{v}}(\boldsymbol{x}, t)$ implies the convergence $\lim_{t\to\infty}\tilde{\boldsymbol{u}}(\boldsymbol{x}, t) = \tilde{\boldsymbol{u}}(\boldsymbol{x})$.

Instead of the stationary solution $\tilde{\boldsymbol{v}}(\boldsymbol{x})$ for (32), we shall investigate properties of the stationary solution $\tilde{\boldsymbol{u}}(\boldsymbol{x})$ for (18), which is a solution to the boundary-value problem

$$M\tilde{\mathfrak{C}}_a(\tilde{\boldsymbol{u}}) = \frac{\partial}{\partial \tilde{u}^a}\tilde{V}(\boldsymbol{\Psi}(\tilde{\boldsymbol{u}})), \tag{36}$$

with the boundary condition $\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{u}_{\mathrm{G}}$ for all $\boldsymbol{x} \in \partial\mathcal{C}_K$. In (36), we have ignored the $o(L^2)$ term.

In order to obtain an insight based on classical mechanics, we assume $g_{ab} = \delta_{ab}$ and $K = 1$. For notational convenience, the superscript 1 is dropped from x^1 for the $K = 1$ case. The differential equation (36) with (20) can be regarded as the Newton equation of motion: The state $\tilde{\boldsymbol{u}}(x)$ is regarded as the position in $\mathcal{D}$ of a free particle with vanishing mass M at time x, moving subject to the inverted potential $-\tilde{V}(\boldsymbol{\Psi}(\tilde{\boldsymbol{u}}))$. Note that x is a temporal variable in this interpretation, whereas it has been introduced as the spatial variable for the SC system. The uniform solution $\tilde{\boldsymbol{u}}(x) = \boldsymbol{u}_{\mathrm{G}}$ corresponds to the situation under which the particle continues to stay at the *unstable* solution $\boldsymbol{u}_{\mathrm{G}}$ of the *inverted* potential $-\tilde{V}(\boldsymbol{\Psi}(\tilde{\boldsymbol{u}}))$. On the other hand, non-uniform solutions to the situation under which the particle at the unstable solution $\boldsymbol{u}_{\mathrm{G}}$ at time $x = -1$ moves somewhere, and comes back to $\boldsymbol{u}_{\mathrm{G}}$ at time $x = 1$. Vanishing mass M implies that the velocity of the particle is infinitely quick. The conservation of mechanical energy

implies that the latter situation never occurs if $\boldsymbol{u}_{\rm G}$ is the unique stable solution that minimizes the potential over all stationary solutions. We follow this intuition to prove the following theorem for any g_{ab} and $K = 1, 2$.

Theorem 2. *Let $K \leq 2$. If the boundary is fixed to the unique stable solution that minimizes the dual potential (5) over all stationary solutions, the uniform solution $\tilde{\boldsymbol{u}}(x) = \boldsymbol{u}_{\rm G}$ is the unique solution to the boundary-value problem (36) with $\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{u}_{\rm G}$ for all $\boldsymbol{x} \in \partial \mathcal{C}_K$ as $M \to 0$.*

We conjecture that it is possible to prove the same conclusion for any K. Theorem 2 implies that, if $\boldsymbol{u}_{\rm G}$ satisfies the assumption in Theorem 2, asymptotic performance of the BP algorithm for the GSC system converges to better performance $P(\boldsymbol{u}_{\rm G})$ for all positions, whereas it for the uncoupled system does to worse performance.

Proof: Let us define the energy tensor $T^{\alpha}_{\beta}(\tilde{\boldsymbol{u}})$ as

$$T^{\alpha}_{\beta}(\tilde{\boldsymbol{u}}) = M\delta^{\alpha\gamma}\frac{\partial \tilde{u}^a}{\partial x^{\gamma}}\frac{\partial \tilde{u}^b}{\partial x^{\beta}}g_{ab}(\tilde{\boldsymbol{u}}) - \delta^{\alpha}_{\beta}\tilde{\mathfrak{L}}\left(\tilde{\boldsymbol{u}}, \frac{\partial \tilde{\boldsymbol{u}}}{\partial \boldsymbol{x}}\right), \quad (37)$$

where the dual Lagrangian $\tilde{\mathfrak{L}}$ is given by

$$\tilde{\mathfrak{L}}\left(\tilde{\boldsymbol{u}}, \frac{\partial \tilde{\boldsymbol{u}}}{\partial \boldsymbol{x}}\right) = \tilde{V}(\boldsymbol{\Psi}(\tilde{\boldsymbol{u}})) + \frac{M}{2}\sum_{\alpha=1}^{K}\frac{\partial \tilde{u}^a}{\partial x^{\alpha}}\frac{\partial \tilde{u}^b}{\partial x^{\alpha}}g_{ab}(\tilde{\boldsymbol{u}}). \quad (38)$$

Since the dual Lagrangian (38) is invariant under the translation of $\boldsymbol{x}$, we use Noether's theorem to find the conservation law

$$T^{\alpha}_{\beta}(\tilde{\boldsymbol{u}}) = -A^{\alpha}_{\beta}(\boldsymbol{x}), \quad (39)$$

where $A^{\alpha}_{\beta}(\boldsymbol{x})$ is a divergence-free tensor independent of $\tilde{\boldsymbol{u}}$, i.e. $\partial A^{\alpha}_{\beta}/\partial x^{\alpha} = 0$. It is straightforward to confirm the conservation law (39) by direct calculation.

First, assume $K = 1$. The conservation law (39) reduces to

$$\frac{M}{2}\frac{d\tilde{u}^a}{dx}\frac{d\tilde{u}^b}{dx}g_{ab}(\tilde{\boldsymbol{u}}) = \tilde{V}(\boldsymbol{\Psi}(\tilde{\boldsymbol{u}})) - A, \quad (40)$$

with a constant A, which corresponds to the conservation of mechanical energy. It is straightforward to find $A = \tilde{V}_{\rm G} = \tilde{V}(\boldsymbol{\Psi}(\boldsymbol{u}_{\rm G}))$ as $M \to 0$, by using the non-negative definiteness of g_{ab}, the boundary condition $\tilde{\boldsymbol{u}}(\pm 1, t) = \boldsymbol{u}_{\rm G}$, and the fact that $d\tilde{\boldsymbol{u}}/dx$ must be infinite for all $\tilde{\boldsymbol{u}} \notin \mathcal{S}$ as $M \to 0$.

Let us prove $\tilde{\boldsymbol{u}}(0) = \boldsymbol{u}_{\rm G}$. Since the differential equation (36) is invariant under the *time* reversal $\tilde{x} = -x$, the stationary solution has the symmetry $\tilde{\boldsymbol{u}}(-x) = \tilde{\boldsymbol{u}}(x)$. Thus, the differentiability of $\tilde{\boldsymbol{u}}$ for x implies $d\tilde{\boldsymbol{u}}(0)/dx = \boldsymbol{0}$. Evaluating (40) at the origin $x = 0$, we find that $\tilde{\boldsymbol{u}}(0) = \boldsymbol{u}_{\rm G}$ must hold from the uniqueness of the stable solution $\boldsymbol{u}_{\rm G}$.

The boundary-value problem on the interval $[-1, 1]$ has been decomposed into two small problems on $[-1, 0]$ and $[0, 1]$. From the time-reversal symmetry of (36), the derivative of the stationary solution for x is zero at the middle point of each interval. This observation and (40) with $A = \tilde{V}_{\rm G}$ imply that $\tilde{\boldsymbol{u}}(x)$ must be equal to $\boldsymbol{u}_{\rm G}$ at the middle points. Repeating this argument, we find that $\tilde{\boldsymbol{u}}(x) = \boldsymbol{u}_{\rm G}$ must hold at countably infinite points. The continuity of the stationary solution implies that $\tilde{\boldsymbol{u}}(x) = \boldsymbol{u}_{\rm G}$ for all $x \in [-1, 1]$ is the unique solution.

Next, assume $K = 2$. The proof is a natural generalization of the $K = 1$ case. We focus on the line segment $\mathcal{L}_1 = \{(x^1, x^2) : -1 \leq x^1 \leq 1, x^2 = 0\} \subset [-1, 1]^2$ of the x^1-axis. Since the differential equation (36) is invariant under the reversal $\tilde{x}^2 = -x^2$, we find the symmetry $\tilde{\boldsymbol{u}}(x^1, -x^2) = \tilde{\boldsymbol{u}}(x^1, x^2)$, which implies $\partial \tilde{\boldsymbol{u}}/\partial x^2 = \boldsymbol{0}$ on the line segment $\mathcal{L}_1$. Thus, the energy tensor (37) is diagonal on the line segment $\mathcal{L}_1$ for $K = 2$, so that A^{α}_{β} is also diagonal on $\mathcal{L}_1$. This implies that $\partial A^{\alpha}_{1}/\partial x^{\alpha} = 0$ reduces to $\partial A^{1}_{1}/\partial x^{1} = 0$. In summary, the $(1, 1)$ element T^1_1 of (37) has the following conservation law

$$\frac{M}{2}\frac{\partial \tilde{u}^a}{\partial x^1}\frac{\partial \tilde{u}^b}{\partial x^1}g_{ab}(\tilde{\boldsymbol{u}}) = \tilde{V}(\boldsymbol{\Psi}(\tilde{\boldsymbol{u}})) - A^1_1 \quad (41)$$

on the line segment $\mathcal{L}_1$, with a constant A^1_1.

We find that $\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{u}_{\rm G}$ must hold on the line segment $\mathcal{L}_1$ by repeating the argument in the proof for $K = 1$. Similarly, we obtain $\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{u}_{\rm G}$ on another line segment $\mathcal{L}_2 = \{(x^1, x^2) : x^1 = 0, -1 \leq x^2 \leq 1\} \subset [-1, 1]^2$. Thus, the boundary-value problem on the square $[-1, 1]^2$ has been decomposed into four small problems. Repeating the argument above implies that $\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{u}_{\rm G}$ must hold on the square $[-1, 1]^2$, since $\tilde{\boldsymbol{u}}(\boldsymbol{x})$ is continuous. ■

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